UDC 534

ON THE PHASE SPACE STRUCTURE AND PERIODIC MOTIONS OF NONAUTONOMOUS DYNAMIC SYSTEMS WITH COLLISION INTERACTIONS

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The method of point mapping is used for investigating the laws governing the phase space structure , and also the conditions of existence and bifurcation of stable periodic motions of nonautomonous dynamic systems with collision interactions. Investigation is based on the identity of phase structures in the proximity of the surface of emergence from the slippage mode region of the simplest nonautonomous system and of systems of a very general form [1, 2]. The results made it possible to formulate a bilateral method for determining the boundaries of regions of slippage, and thus provide the final proof of the iterative computation procedure [2 — 4]. The method developed here is illustrated on the extension of the model used for defining various vibro-shock mechanisms [5].

1. Motion of a material point in a one-dimensional half-space subjected to an external linear force. The considered model represents the simplest system with collision interactions. The subsequent analysis will show that it brings out in the small the singularities of dynamics of a fairly wide class of systems.

1. 1. The equations of motion are of the form

$$y^{\cdot\cdot} = t, \quad y > \lambda_0 \tag{1.1}$$

$$y(t_k) = \lambda_0, \quad y^{\cdot}(t_k + 0) = -Ry_k^{\cdot}, \quad y_k^{\cdot} \equiv y^{\cdot}(t_k - 0) < 0 \tag{1.2}$$

$$y = \lambda_0, \; y^{\cdot} = 0, \; t < 0 \quad (t = (F\tau - Q) / m, \; y = (F / m)^2 \eta) \tag{1.3}$$

where the variables t (independent) and y are related to the dimensional quantities τ (time) and η (the distance of a material point from a fixed reference point O) by the above formulas; m is the mass of the material point; $F > 0$ and $Q \geqslant 0$ are parameters of the external force $F\tau - Q$ that is linear with respect to time; $R \in [0, 1]$ is Newton's coefficient of velocity recovery after a collision, and λ_0 is the displacement of O from the half-space boundary (the limiter). Usually the quantity λ_0 is made nonnegative [2, 4, 6], however in this Section it is sufficient to set $\lambda_0 = 0$.

The phase space Φ of system (1. 1) — (1. 3) is defined by the coordinates $t, y \geqslant 0$, $y^{\cdot}$, hence it is three-dimensional ($\dim \Phi = 3$). In the case of piecewise continuous systems it is sufficient to subdivide the phase space only at the surface of merging of trajectory sections on Π $(y = 0)$ [6]; in our case even on a part of it only, namely on the half-plane Π_- $(y = 0, \; y^{\cdot} < 0)$ or Π_+ $(y = 0, y^{\cdot} > 0)$. The subdivision is carried out by the method of point mapping. For this we introduce the inverse mapping T^- of Π_- onto itself; it is generated by the trajectories (1. 1) and (1. 2) between the k-th and the $(k + 1)$-th collisions $(t_k > t_{k+1})$. Formulas for T^- are obtained by integrating (1. 1) for initial conditions (1. 2). They are

$$t_{k+1} = t_k - \sigma_k, \quad \dot{y}_{k+1} = (4y_k^{\cdot} - \sigma_k t_k) / 2R \qquad (1.4)$$
$$\sigma_k = -3t_k / 2 + [(3t_k / 2)^2 - 6y_k^{\cdot}]^{1/2}$$

The last equality implies the existence of the point map T^- throughout Π. Let us define the kind of dependence of T^- on coordinates of the "reference" point M_k (t_k, $y_k = 0$, $y^{\cdot}_k$) when the latter tends to the boundary Γ ($y = 0$, $y^{\cdot} = 0$) between Π_- and Π_+. For this we define the limit values of expressions for $y^{\cdot}_{k+1}$ and σ_k as follows:

$$\lim_{y_k^{\cdot} \to 0} \dot{y}_{k+1} = -\sigma_k t_k / 2R, \quad \lim_{y_k^{\cdot} \to 0} \sigma_k = 3t_k (1 + \operatorname{sign} t_k) / 2 \qquad (1.5)$$

It is obvious from (1.5) that the passing to limit Γ in the quarter-plane Π_-^- ($t < 0$, $y = 0$, $y^{\cdot} < 0$) results in the degeneration of map T^- into an identical one, while in region Π_-^+ ($t > 0$, $y = 0$, $y^{\cdot} < 0$) the map does not degenerate. At the boundary point M° between the semiaxes Γ_+ ($t > 0$, $y = 0$, $y^{\cdot} = 0$) and Γ_- ($t < 0$, $y = 0$, $y^{\cdot} = 0$) the mapping T^- converts continuously to an identical one. The described properties of passing to limit make it possible to uniquely determine T^- at the boundary of half-plane Π_- by using formulas (1.5) and (1.3) for Γ_+ and $\Gamma \setminus \Gamma_+$, respectively.

We apply to Π_- transformation T^{-k} which is the k-fold product of maps T^-. We then obtain region $\Pi_k \equiv T^{-k}\Pi_- \subset \Pi_-$ whose boundary is evidently determined as the image of transformation $T^{-k}\Gamma$, i.e. it consists of semiaxis Γ_- and curve $\Gamma_k \equiv T^{-k}\Gamma_+$, adjacent to M°. With the use of Eqs. (1.4) it is possible to prove by induction that $\Gamma_k \subset \Pi_-^-$ and that it represents the semiparabola

$$y^{\cdot} = -\gamma_k t^2, \quad \gamma_k > 0, \quad t < 0 \qquad (1.6)$$

where the coefficients $\gamma_k = \gamma_k(R)$ and $\gamma_{k+1} = \gamma_{k+1}(R)$ are related to each other by the parameter $\Theta_k \equiv -\sigma_k / t_{k+1}$ and by the following formulas:

$$\gamma_k = \frac{3\Theta_k - 2\Theta_k^2}{6(1-\Theta_k)^2}, \quad \gamma_{k+1} = \frac{3\Theta_k - \Theta_k^2}{6R}, \quad \gamma_1 = \frac{3}{8R} \qquad (1.7)$$

Since $t_k \in [t_{k+1}, 0]$, hence

$$\Theta_k \in [0, 1] \qquad (1.8)$$

When k is run through a sequence of positive integers the ordered sequence $\{\Gamma_k\}$ must develop in a positive or negative direction (see Fig. 1, a, where the curve denoted by k corresponds to Γ_k), if the Jacobian of map T^- in Π_- retains its sign. Computing the partial derivatives $\partial \dot{y}_{k+1} / \partial y_k^{\cdot}$, $\partial \dot{y}_{k+1} / \partial t_k$, $\partial t_{k+1} / \partial y_k^{\cdot}$, and $\partial t_{k+1} / \partial t_k$ and expanding the determinant $\partial (y_{k+1}, t_{k+1}) / \partial (y_k^{\cdot}, t_k)$, in accordance with (1,8), we obtain

$$J = (3 - 2\Theta_k) / R (3 - \Theta_k) > 0 \qquad (1.9)$$

Since semiparabola Γ_1 is turned relative to Γ_+ in the positive direction, inequality (1.9) ensures the same orientation of Γ_{k+1} relative to Γ_k, hence $\gamma_{k+1} < \gamma_k$ [7]. Note that on Γ_+ we have $J = 0$. However, owing to the continuity of T^-, this degeneration is of no importance for the determination of orientation of Γ_k and Γ_{k+1} relative to each other.

Thus the bounded numerical sequence $\{\gamma_k\}$ monotonically decreases with γ_s as its limit. The expression for the latter in terms of R is obtained from (1.7) and the supplementary condition

$$\gamma_k = \gamma_{k+1} = \gamma_s \qquad (1.10)$$

After some simplifying transformations, for the coefficient γ_s of semiparabola Γ_s, which is invariant with respect to T^-, we obtain the relationship

$$\gamma_s = (3\Theta_s - \Theta_s^2) / 6R, \quad \Theta_s^3 - 5Q_s^2 + (7-2R)\Theta_s - 3(1-R) = 0 \quad (1.11)$$

where the symbol Θ_s denotes values of Θ_k segregated by (1.10).

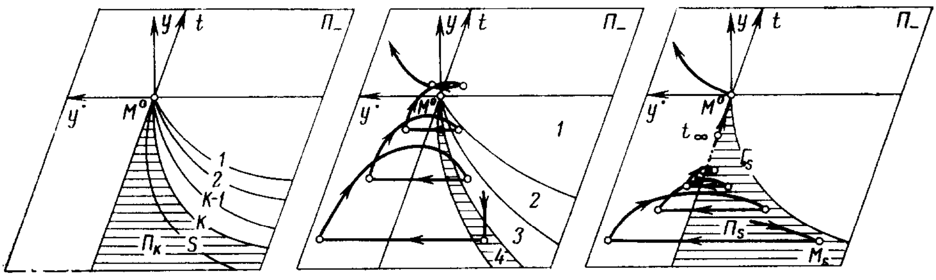

Fig. 1

By the Sturm theorem [8] only one of the three real roots of the second of Eqs. (1.11) satisfies condition $\Theta_s \in [0, 1]$. That root monotonically decreases in the interval [1, 0], when R increases from 0 to 1.

With the use of the transformation $\Theta_s = 1 - \lambda$ it is possible to obtain from the second of formulas (1.11) the characteristic equation [9, 10] and, thus to relate formally parameter λ to the quantity Θ_s whose physical meaning is clearly that of the coefficient of the slippage mode duration [2]. It also indicates the feasibility of obtaining the equation of semiparabola Γ_s by the method developed by R. F. Nagaev.

Thus the sequence of semiparabolas $\{\Gamma_k\}$ and Γ_s divides Π_- into a denumerable set of regions

$$\Pi_- \supset \Pi_1 \supset \ldots \supset \Pi_k \supset \Pi_{k+1} \supset \ldots \supset \Pi_s \equiv \lim_{k\to\infty} \Pi_k$$

embedded in one another.

The evident reversibility of T^- clearly implies that when $M_k \in P_k = \Pi_{k-1} \setminus \Pi_k$, the motion of system (1.1) — (1.3) is accompanied by k collision interactions that consecutively occur in subregions $P_k, P_{k-1}, \ldots, P_1$. The mapping point $M(t, y, y^\cdot)$ having passed through P_1, moves into the half-space $G\ (y > 0)$ and remains there in the interval (t_1, ∞) (see Fig. 1, b where the shaded region $k = 4$ corresponds to P_4).

The shift of M_k onto the boundary of subregion P_k leads either to the degeneration of the final collision into a contact $(M_k \to \Gamma_{k-1})$, or to an additional interaction of a kind of contact between trajectory (1.1) and surface Π $(M_k \to \Gamma_k)$ which takes place at some instant $t_c \in (t_1, \infty)$. If the initial conditions relate to region Π_s bounded by Γ_- and Γ_s, a slippage mode is realized in the system [1 — 4]. In fact, by definition the mapping point returns in this case to Π_- after each collision and shifts at the same time in the direction of increasing t. Thus, after a denumerable set of transitions along collision-collisionless sections of trajectories, point M at some limiting instant of time $t_\infty < 0$ reaches the semiaxis Γ , and then moves along it in accordance with the law (1.3) until it arrives at the convergence point M° (Fig. 1, c). The duration h of an infinite-collision process originating at point $M_s(t_s, 0, y_s^\cdot) \in \Pi_s$, is obviously finite and equal to the difference $t_\infty - t_s$. The value $h = -t_s$ relates to the border process that develops along Γ_s and decreases to zero with decreasing R. The general asymptotic representation of quantity h was obtained in [2].

Since $y^{\cdots}(t)|_0 > 0$, the mapping point moves from M° along the trajectory (1.1) to G, where in the interval $(0, \infty)$ it monotonically moves away from Π. Note that when $F < 0$, $\Pi_s = \Pi_-$ and $y^{\cdots}(t) < 0$, which means that M after reaching at instant t_∞ the semiaxis Γ_-, continues to move along it in the interval (t_∞, ∞).

Thus the system of configurations of $\{P_k\}$ and Π_s represents the complete subdivision of Π_- into regions where the material point moves in a one-dimensional half-space with different numbers of collisions with its boundary.

Let us examine the character of displacement and deformations of P_k and Π_s produced by variation of the system parameter. Formulas (1.7) and (1.11) imply that coefficients γ_k and γ_s monotonically decrease in the interval [0, 1] from infinity to $\gamma_k(1) > 0$ and $\gamma_s(1) = 0$, respectively (see Fig. 2, where curves 1, 2 and 3 correspond to γ_1, γ_2 and γ_s). Semiparabolas Γ_k and Γ_s are, consequently, turned from the semiaxis $Y_-(t = 0,\ y = 0,\ y^\cdot < 0)$ in the positive direction up to the limit position defined by coefficients $\gamma_k(1)$ and $\gamma_s(1)$. In that case subregions P_k and Π_s behave as described below.

Owing to the displacement of Γ_1 to the position corresponding to $\gamma_1 = {}^3/_8$, P_1 widens in the direction away from the boundaries of Π_-^+ .

For $R = +0$, P_k $(k = 2, 3, \ldots)$ are generated as flat regions from the one-dimensional set Y_- , and then rotated around M° in the positive direction. Simultaneously, as shown by the analysis of $d\gamma_{k+1} / d\gamma_k$, P_k first widens and then contracts.

For $R = 0$ we have $\Pi_s = \Pi_-^-$. With increasing R region Π_s sharply contracts owing to the displacement of Γ_s , and for $R = 1$ degenerates into the semiaxis Γ_-. Examples of subdivision of Π_- are shown in Fig. 3. The semiparabolas Γ_1, Γ_2 and Γ_s represented by solid lines (curves 1, 2 and 3, respectively) relate to $R = 0.2$, while those shown by dash and dash-dot lines relate, respectively, to $R = 0.5$ and $R = 0.75$.

If R fluctuates, the established above extremely fine phase structure is to a considerable extent disrupted. For instance, if the maximum fluctuation range is $\Delta R = 0.05$, the inequality $\gamma_2(R + \Delta R) \leqslant \gamma_s(R - \Delta R)$ is satisfied in the interval $R \in [0, 0.4]$ (Fig. 2) and, consequently, the set $\{P_k\}$ $(k = 3, 4, \ldots)$ is completely covered by parts of P_2 and Π_s configurations. If $R \in [0, 0.25]$, then $\gamma_1(R + \Delta R) \leqslant \gamma_s(R - \Delta R)$ and the system of overlapping subregions comprises also P_2. In that case the semiparabolas with coefficients $\gamma_1(R - \Delta R)$ and $\gamma_s(R + \Delta R)$ subdivide half-plane Π_- into three basic regions: P_1^*, P^* and Π_s^*. It can be reliably assumed that P_1^* and Π_s^* correspond to single-collision and slippage modes, respectively. In subregion P^*, which lies between P_1^* and Π_s^*, motions with any number k of collisions may take place. The probability of considerable values of k is the higher the farther the initial point in P^* lies from the parabola with coefficient $\gamma_1(R - \Delta R)$.

Owing to unavoidable errors of idealization and to fluctuations of R , a separate consideration of k-collision motions in actual computations is expedient only for $k \leqslant k^*$, where k^* depends on ΔR and other (negligible) errors. If $k > k^*$ it is reasonable to relate in the dynamic model the denumerable set of subregions P_k either to P_{k*} or to Π_s. Such approximation leads essentially to a refinement of the model which is equivalent to (1.1) — (1.3) but has an incomparably simpler pattern of subdivision of Π_-. Usually $k^* \leqslant 3$ [2 — 4], and in the simplest case $\Pi_s \supset P_k$ $(k > 1)$ [11].

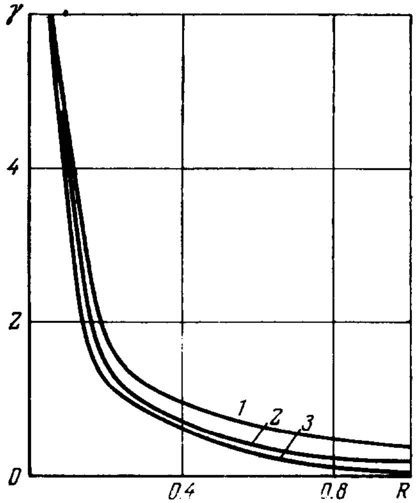

Fig. 2

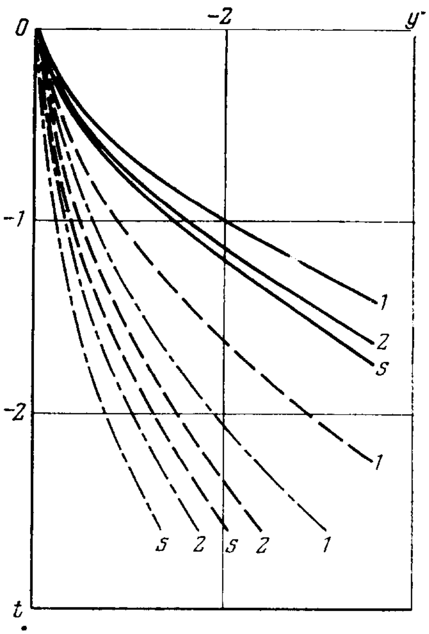

Fig. 3

1.2. Formulas (1.7) after transformation of the first of these to

$$\Theta_k = [3\,(1 + 4\gamma_k) - (9 + 24\gamma_k)^{1/2}\,] \,/\, 4\,(1 + 3\gamma_k)$$

can be formally treated as defining an iterative process for solving the problem of separating the boundary of region Π_s. Let us consider in connection with that problem the properties of solutions which are obtained in the above investigations and make it possible to formulate a general method of deriving successive approximations that converge to Γ_s.

First, let us note that, as implied by (1.7), map T^- converts any arbitrary semiparabola $\Gamma' \subset \Pi_-$ that originates at M° to semiparabola $\Gamma_1' \equiv T^-\Gamma' \subset \Pi_-^-$. Hence, taking into consideration the uniqueness of the invariant curve Γ_s and condition (1.9), it is possible to assert that the iteration sequence

$$\{\Gamma_k' \equiv T^{-k}\Gamma'\} \quad (k = 1, 2, \ldots) \tag{1.12}$$

converges to Γ_s and monotonically develops in a fixed direction which is negative inside Π_s when $\Gamma' \subset \Pi_s$, or positive outside Π_s in the opposite case. The described features of (1.12) are represented in the Koenig-Lamery diagram constructed with the use of (1.7) (Fig. 4).

Let us now take the next step. Since the plane subregion comprised between Γ_{k-1}' and Γ_k' degenerates for $k \to \infty$ into a one-dimensional set Γ_s, the iteration (1.12), where Γ' is an arbitrary continuous curve that originates at M°, always converges to the boundary of region Π_s. In other words, the map T^- in the space of curves (1.12) metrized in some way is pressing throughout the region toward the invariant semiparabola Γ_s. The direct mapping T $(t_k < t_{k+1})$ is pressing toward Γ^- in the space of curves $\Gamma_{-k}' \subset \Pi_s$, i.e. in its own part of the region of determination of Π_1.

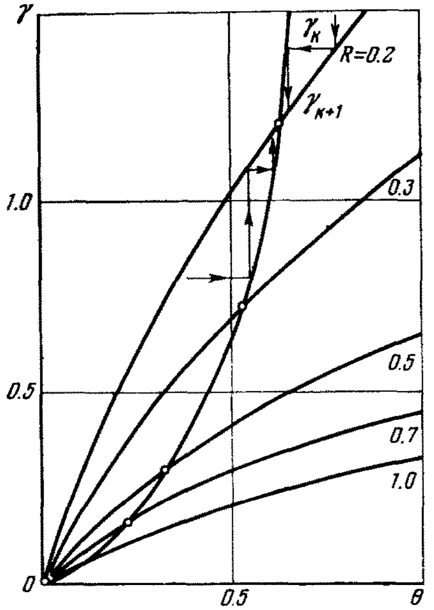

Fig. 4

For the actual determination of Γ_s by means of (1. 12) it is evidently sufficient to take as Γ' the ε-section adjacent to the convergence point of an arbitrary curve that is continuous in an as small as desired ε-neighborhood of M°.

Note 1. 1. The computation process (1. 12) is directly applicable to the solution of the general problem of separating the boundary Γ_{si} of region Π_{si}, which is determined by the instant of completion of the infinite-collision process and does not exceed a specified $t_i < 0$. In such case the ε-section adjacent to M_i $(t_i, 0, 0)$ is taken as the initial approximation.

2. The dynamic system with a single degree of freedom (general case). The system of the very general form considered in this Section may be interpreted as the law of motion of a point of mass m in a one-dimensional half-plane subjected to the action of an arbitrary external force.

2. 1. Motion of the point in the time intervals $(t_k,\ t_{k+1})$ is defined by formulas

$$y^{\cdot\cdot} = f(t,\ y,\ y^\cdot), \quad y > \lambda_0 \tag{2. 1}$$

Collision interactions in the case of a limiter are subject to law (1. 2). Functior f $(t,$ $y,$ $y^\cdot)$ represents in Eq. (2. 1) the total force interaction of the external medium and excitation forces normalized with respect to m. We assume that $f \in C^2$ and depends on parameters $\lambda_1, \ldots, \lambda_r$ (r is a positive integer) which together with λ_0 and R constitute space D $(\dim D = r + 2)$.

As in the case of (1. 1) — (1. 3), the phase space Φ of system (1. 2), (2. 1), (2. 2) is defined by the coordinates $t,\ y \geqslant \lambda_0,\ y^\cdot$ and $\dim \Phi = 3$. If during its motion point M reaches section Γ_- $(y = \lambda_0,\ y^\cdot = 0,\ y^{\cdot\cdot} < 0)$, its subsequent behavior is no longer determined by Eqs. (1. 2) and (2. 1) and must be additionally defined. For this we use the equations of kinematics

$$y = \lambda_0,\ y^\cdot = 0,\ f < 0 \tag{2. 2}$$

Formula (2. 2) defines the motion of the phase point along Γ_- up to the instant when it reaches the state M°

$$y^\circ = \lambda_0, \quad y^{\cdot\circ} = 0, \quad f|_{M^\circ} = 0, \quad \left.\frac{\partial f}{\partial t}\right|_{M^\circ} > 0 \tag{2. 3}$$

from which M moves to G along the trajectory (2. 1).

In what follows we assume that $t^\circ = 0$. If a real solution of (2. 3) exists, such equality can be always obtained by a linear substitution of the variable t.

The additional definition thus derived is physically sound. Formally it presupposes the existence in the small neighborhood of Γ_- of map T (T^-) of region Π_- $(y = \lambda_0,$ $y^{\cdot} < 0)$ onto itself generated by relationships (1. 2) and (2. 1), and also of the continuity of conversion of T (T^-) to an identical transformation when the initial point M_k tends to Γ^-. The indicated features of T (T^-) are determined by the properties of f which in specific systems can be established by a conventional test. In the small neighborhood of M° the properties of $T\,(T^-)$ mapping imposed by (2. 2) follow from the topological identity (which is proved below) of phase structures of systems (1. 1) — (1. 3) and (1. 2), (2. 1), (2. 2) in the neighborhood of the convergence point.

To prove this we use Taylor's multiple formula and set, without loss of generality, $\lambda_0 = 0$. Taking into account (2. 3), we write

$$f = \frac{\partial f}{\partial t}\bigg|_{M^\circ} t + \frac{\partial f}{\partial y}\bigg|_{M^\circ} y + \frac{\partial f}{\partial y^{\cdot}}\bigg|_{M^\circ} y^{\cdot} + \chi\,(t', y', y^{\cdot\prime}) \qquad (2.4)$$

$$t' \in (0,\ t),\ y' \in (0,\ y),\ y^{\cdot\prime} \in (0,\ y^{\cdot})$$

where χ is the residual term.

Assuming for the time being that $(\partial f\,/\,\partial t \cdot df\,/\,\partial y \cdot df\,/\,\partial y^{\cdot})\,|_{M^\circ} \neq 0$, we find that the behavior of trajectories (2. 1) in proximity of M° is defined by the following linearized equation of collisionless motions:

$$y^{\cdot\cdot} = \frac{\partial f}{\partial t}\bigg|_{M^\circ} t + \frac{\partial f}{\partial y}\bigg|_{M^\circ} y + \frac{\partial f}{\partial y^{\cdot}}\bigg|_{M^\circ} y^{\cdot} \qquad (2.5)$$

To estimate the zero solution of (2. 5) in the neighborhood of M° we represent $y\,(t)$ in the form of Maclaurin's series. This with (2. 3) yields the relationships $y\,(t) = O\,(t^3)$ and $y^{\cdot}\,(t) = O\,(t^2)$.

The behavior of trajectories (2. 1) in the neighborhood of the convergence point is essentially determined by the nonautonomous term $(\partial f\,/\,\partial t)_{M^\circ}\, t$ of expansion (2. 4). The remaining terms and, consequently, also condition $(\partial f\,/\,\partial y \cdot \partial t\,/\,\partial y^{\cdot})_{M^\circ} \neq 0$ can be neglected. We assume that $(\partial f\,/\,\partial t)_{M^\circ} = 1$ to within the scale of the independent variable. This yields the system considered in Sect. 1 and, evidently, proves the topological identity of phase structures of (1. 1) — (1. 3) and (1. 2), (2. 1), (2. 2) in the neighborhood of M°.

The subdivision of Π_- as a whole is determined by the derived specific sequence $\{\Gamma_k \equiv T^{-k}\Gamma_+\}$, where Γ_+ is specified by the relationships

$$y = \lambda_0,\ \ y^{\cdot} = 0,\ \ y^{\cdot\cdot} > 0 \qquad (2.6)$$

It is, however, evident that in the neighborhood of the convergence point the sequence $\{\Gamma_k\}$ coincides within smalls of higher order with the pencil of semiparabolas (1. 6).

In a number of practically important cases $f = \varphi\,(t) - \omega^2 y - \nu y^{\cdot}$ [2 — 6]. For such models the Jacobian of T mapping is of the form $J = \exp\,(\nu\sigma_k)\, y_k^{\cdot} / R^2 y_{k+1} > 0$, hence the phase structures of systems (1. 1) — (1. 3) and (1. 2), (2. 1), (2. 2) are topologically identical not only locally but throughout Π_-

2. 2. The above analysis shows that the iteration process (1. 12), where Γ' is the ε-section adjacent to the convergence point, always converges to Γ_s of region Π_s of system (1. 2), (2. 1), (2. 2). In computing specific models it is convenient to use the bilateral procedure (1. 12), since it makes possible to estimate the closeness of the k-th iteration to Γ_s at each k-th step, using, for example, the integral estimate of δ of the form

$$\delta(t) = \left| \int_t^0 [\Psi_{k+}(t) - \Psi_{k-}(t)]\, dt \right|$$

where Ψ_{k+} and Ψ_{k-} define the variation of $y^{\cdot}$ along the curves of internal $\{\Gamma_{k+}' \equiv T^{-n}\Gamma_+' \subset \Pi_s\}$ or external $\{\Gamma_{k-} \equiv T^{-k}\Gamma_-' \equiv \Pi_s\}$ iteration sequences; $\Gamma_+' \subset \Pi_s$ and $\Gamma_-' \equiv \Pi_s$ are the related starting ε-sections.

Theoretically the absolute convergence of the computation process (1.12) differs from successive approximations to Γ_s in the form of series in powers of some small parameter [9, 10]. Such approximations have any meaning only inside the convergence circle of the resolving series. The determination of the radius of such circle is extremely complicated even for the simplest systems. As far as the author is aware, this problem was not considered in any works on the dynamics of systems with collisions.

2.3. The case when function f is 2π-periodic with respect to t is particularly interesting, since it admits the existence of stable periodic motions. If we assume $\lambda_0 > 0$, then, owing to the identity of planes $t = 2n\pi$ (n is an integer), space Φ can be considered to be a region consisting of points lying on and outside a cylindrical surface Π of radius λ_0. Since in such representation of Φ the coordinate t is taken along the directing circumference Γ ($y = \lambda_0$, $y^{\cdot} = 0$) from a certain fixed radial direction, it is clear that j-fold (j is a positive integer) periodic motions are defined by trajectories (1.2), (2.1) and (2.2) which envelop Π j-times (see Fig. 5, a, where $j = 3$). If a k-collision periodic motion takes place in the system, a k-term cycle of stationary points $M_{k-i} \equiv T^i M_k \subset P_{k-i}$ ($i = 0$ for mod k) of map T (Fig. 5, b, $j = 1$) corresponds to it. When the stationary points pass from inside P_{k-i} either on Γ_{k-i} or Γ_{k-i-1}, a C-bifurcation of the k-collision motion takes place. If $M_* \equiv TM^{\circ} \in \Pi_s$ (Fig. 5, c, $j = 1$) a periodic motion with a section of slippage mode obtains in the system (1.2), (2.1), (2.2). Its period L is defined as the time interval between two consecutive passages of the mapping point through M°, and its multiplicity is the maximum number j which ensures the equality $L = 0$ for mod $2j\pi$. Since $\dim M^{\circ} = 0$, the periodic motions of (1.2), (2.1), (2.2) which comprise the slippage mode section are always stable, and under external effects which do not violate the conditions of existence, the time of complete recovery of such oscillations is lower than L.

Bifuractions of periodic motions with slippage mode section take place in the following cases:

a) The slippage mode develops on Γ_s. The set of parameter values that correspond to such degeneration form in D the boundary C_s of the region of existence and stability. They are determined by the relationship $M_* \in \Gamma_s$.

b) The real solution of (2.4) vanishes. A cylindrical N-surface independent of R, whose equations are obtained from (2.3) by the substitution of condition $(\partial f / \partial t)|_{M^{\circ}} = 0$ for the inequality, corresponds to this case in space D.

c) At some instant $t_c \in (0, t_*)$ the trajectories (2.1) of configuration Π become tangent. The cylindrical C-boundary independent of R in D along which appears the indicated degeneration is determined by the analytic conditions of tangency

$$y(t)|_{t_c} = 0, \quad y^{\cdot}(t)|_{t_c} = 0$$

d) The hypersurfaces C_0 ($R = 0$) and C_1 ($R = 1$) correspond to the limits of the range of physical values of the coefficient of velocity recovery. The segment (t_*, t_∞) contracts at points C_0 to a point, while along C_1 the plane region Π_s degenerates into one-dimensional set Γ_-.

Note 2.1. Generally speaking system (2.3) can have $l \in [1, 2, \ldots, \infty)$ different solutions. To each of these corresponds its own region of slippage modes Π_{sq} ($q \in [1, 2, \ldots, l]$). In such case the periodic motions may contain q sections of slippage modes.

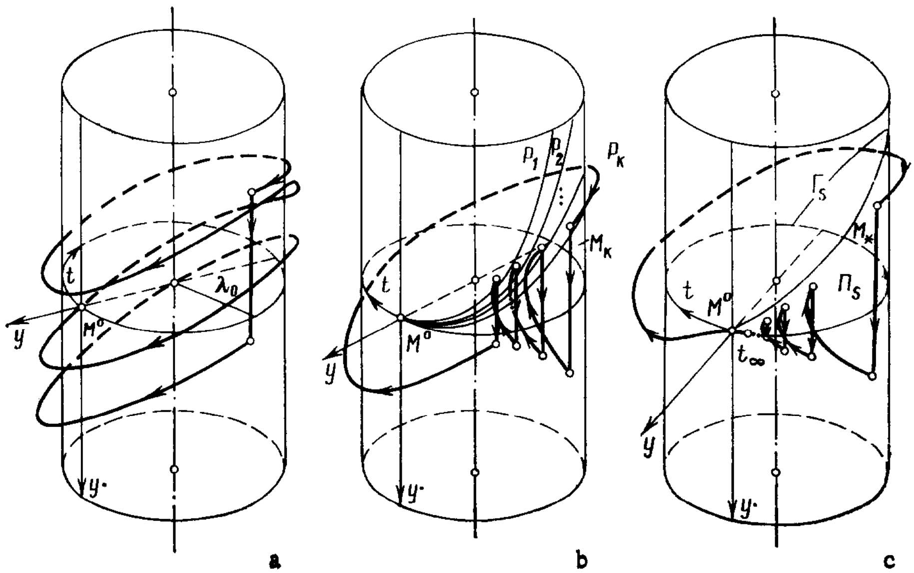

Fig. 5

Note 2.2. At points of sequence $\{M_m^\circ\}$ ($m \in [1, 2, \ldots, \infty)$) function $f(t, \lambda_0, 0)$ suffer discontinuities of the kind of a finite shock, and at points of some sequence $\{M_p^\circ\} \in \{M_m^\circ\}$ the inequality $f(t_p^\circ - 0, \lambda_0, 0) < 0 < f(t_p^\circ + 0, \lambda_0, 0)$ is valid. Then in the neighborhood of every point M_p° there exists a region of slippage modes Π_{sp}. When constructing the boundary Γ_{sp} with the use of (1.12), the ε-section adjacent to the convergence point M_p° in the interval $t \in (t_p^\circ - \varepsilon, t_p^\circ)$ is to be taken as the input approximation.

Example. In the case of a single-mass collision-oscillating system without natural frequency ($\omega = 0$) but with viscous friction force present ($\nu > 0$) function $\varphi(t) = \text{const} - Q$ ($Q > 0$). Integrating Eq. (2.1) with allowance for the above and (1.2), for the T^- mapping of the half-plane Π_- onto itself, we obtain the following system of equations:

$$t_{k+1} = t_k - \sigma_k,\ \dot{y}_{k+1} = R^{-1}\{Q\nu^{-1} - \alpha\,[\nu^1 \cos(t_k - \sigma_k) + \sin(t_k - \sigma_k)] -$$
$$\exp(\nu\sigma_k)\,[y_k^{\cdot} + Q\nu^{-1} - \alpha(\nu\cos t_k + \sin t_k)]\}$$

where σ_k is the smallest positive root of equation

$$[\exp(\nu\sigma_k) - 1]\,[\alpha(\nu\cos t_k + \sin t_k) - Q\nu^{-1} - y_k^{\cdot}] + \alpha\nu\,\{\cos t_k -$$
$$\cos(t_k - \sigma_k) + \nu\,[\sin(t_k - \sigma_k) - \sin t_k]\} + Q\sigma_k = 0,\ \alpha = (\nu^2 + 1)^{-1}$$

Applying the T^{-k} mapping to section Γ_+ ($t \in (-\arccos Q, \arccos Q), 0, 0$), we obtain the subdivision of the half-plane Π_-. This is represented in Fig. 6 for $Q = 0.1$, $\nu = 0.5$, $R = 0.2$ (solid lines), and $R = 0.3$ (dash lines); curves 1, 2 and s relate to

Γ_1, Γ_2 and Γ_8. The relative position of curves shows that elements of sequence $\{\Gamma_k\}$ are close to Γ_8 already for $k > 2$. To determine more precisely the boundary Γ_8 we use Eq. (1.12). The input approximation $\Gamma' \in \Pi_8$ was determined by the general sufficiency conditions [1, 3].

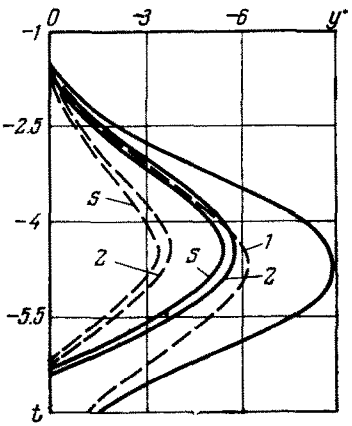

Fig. 6

3. The system with n degrees of freedom. Let us consider as a reasonable extension of model (1.2), (2.1), (2.2) the system with an arbitrary finite number n of degrees of freedom which contains a fixed pair of colliding elements.

Collisionless motions of a model of fairly general form are defined by the following equations:

$$y^{\cdot\cdot} = f_0(t, y, y^{\cdot}, x_1, \ldots, x_{2(n-1)}), \quad y > \lambda_0 \quad (3.1)$$

$$x_i^{\cdot} = f_i(t, y, y^{\cdot}, x_1, \ldots, x_{2(n-1)}) \quad (3.2)$$

$$(i = 1, 2, \ldots, 2(n-1))$$

Collisions between interacting elements are subject to the law (1.2). The discontinuities of derivatives $x_i^{\cdot}$ at instants t_k are uniquely defined by (3.2) as the differences between the values f_i $(i = 1, 2, \ldots, 2(n-1))$ at instants $t_k - 0$ and $t_k + 0$.

The dimensionless relative displacement of elements of the collision pair y and the generalized coordinates $x_1, \ldots, x_{2(n-1)}$ with the variables t and $y^{\cdot}$ constitute the phase space Φ $(\dim \Phi = 2n + 1)$.

We assume that functions $f_i \in C^2$ $(i = 0, 1, \ldots, 2(n-1))$, and depend on parameters $\lambda_1, \ldots, \lambda_r$ which together with λ_0 and R constitute space D $(\dim D = r + 2)$.

Passing to the investigation of the behavior of phase trajectories (3.1) in the neighborhood of the $2(n-1)$-dimensional surface M° which is defined by formulas (2.3), we note that it is possible to show, as in Sect. 2, the validity of estimates

$$y(t) = \lambda_0 + O((t - t^{\circ})^3), \quad y^{\cdot}(t) = O((t - t^{\circ})^2)$$

Hence, if

$$\prod_{i=0}^{2(n-1)} (\partial f_i / \partial t)|_{M^{\circ}} \neq 0 \quad \text{or} \quad \prod_{i=0}^{2(n-1)} (\partial f_i / \partial x_j \cdot x_i)|_{M^{\circ}} \neq 0$$

then in the small neighborhood of M° the dependence of f_i $(i = 0, 1, \ldots, 2(n-1))$ on y and $y^{\cdot}$ can be neglected, and the investigation of (3.1) and (3.2) may be carried out with the use of the following separated system of equations:

$$y^{\cdot\cdot} = f_0(t, \lambda_0, 0, x_1, \ldots, x_{2(n-1)}), \quad y > \lambda_0 \quad (3.3)$$

$$x_i^{\cdot} = f_i(t, \lambda_0, 0, x_1, \ldots, x_{2(n-1)}) \quad (i = 1, 2, \ldots, 2(n-1)) \quad (3.4)$$

By integrating (if only numerically) (3.4) with initial conditions disposed on M° and substituting the result into (3.3), we obtain the relationship

$$y^{\cdot\cdot} = f(t, \lambda_0, t^{\circ}, x_1^{\circ}, \ldots, \dot{x}_{2(n-1)}), \quad y > \lambda_0$$

which may be considered to be the law of collisionless motions of a material point in a one-dimensional half-space under the action of force f, which depends on "parameters" t°, x_1°, . . . , $x_{2(n-1)}$. These parameters are obviously subject to the restriction (2.3). Using the results obtained in Sect. 2, we complete the definition of the behavior of a system with n degrees of freedom by equations of kinematics in the case when the phase point reaches the $(2n-1)$-dimensional half-surface Γ_- $(y = \lambda_0,\ y^\cdot = 0,\ y^{\cdot\cdot} < 0)$ ending in (2.3) by the escape of trajectories (3.1) and (3.2) into G. It is also possible to state that the behavior of phase trajectories of system (1.2), (3.1), (3.2) in the neighborhood of M°, as well as the subdivision structure of half-space Π_- $(y = \lambda_0,\ y^\cdot < 0)$ are determined (the latter within the dimensional accuracy) by the simplest model defined by (1.1) — (1.3). The character of subdivision of Π_- varies quantitatively as a whole depending on the specific realization of the sequence $(2n-1)$-dimensional surfaces $\Gamma_k \equiv T^{-k}\Gamma_+$, where Γ_+ is also a $(2n-1)$-dimensional surface isolated by condition (2.6).

All conclusions reached in Sect. 2 about the iteration procedure for determining the limit surface Γ_s, disposition of stationary points of map T^k, and also about the conditions of their existence and bifurcations, are valid in the general case considered here. The theoretical difference between multidimensional and one-dimensional systems consists in that regions of existence of periodic motions containing a section of slippage mode do not, generally, coincide with regions of their stability (owing to the relationship $\dim M^\circ = 2(n-1) \neq 0$).

We note in conclusion that for actual separation of Γ_s for the sake of reducing the number of iterations that would ensure the specified accuracy, it is possible, in the case of systems of dimension $n > 1$, to take as the input approximation the ε-section which is isolated by intersepts of series in powers of a small parameter [9, 10].

REFERENCES

1. Feigin, M. I., Slippage in dynamic systems with collision interactions. PMM, Vol. 31, № 3, 1967.
2. Fedosenko, Iu. S. and Feigin, M. I., On the theory of slippage state in dynamical systems with collisions. PMM Vol. 36, № 5, 1972.
3. Feigin, M. I., Special feature of dynamics of systems with collision interactions connected with the existence of slippage motions. Collection: Mekhanika Mashin, N^{os} 33 and 34, Moscow, "Nauka", 1972.
4. Fedosenko, Iu. S. and Feigin, M. I., Periodic motions of a vibrating striker including a slippage region. PMM Vol. 35, № 5, 1971.
5. Kobrinskii, A. E. and Kobrinskii, A. A., Vibro-impact Systems. Moscow, "Nauka", 1973.
6. Bespalova, L. V., Neimark, Iu. I. and Feigin, M. I., Dynamic systems with collision interactions and the theory of nonlinear oscillations. Inzh. Zh. MTT, № 1, 1966.
7. Bermant, A. F., Mapping, Curvilinear Coordinates. Moscow, Fizmatgiz, 1958.
8. Korn, G. and Korn, T., Mathematics Reference Book for Scientists and Engineers. Moscow, "Nauka", 1970.
9. Nagaev, R. F., On the analytic definition of a quasi-plastic impact. Izv. Akad. Nauk SSSR, MTT, № 4, 1970.
10. Nagaev, R. F., The general problem of the quasi-plastic impact. Izv. Akad. Nauk SSSR, MTT, №3, 1971.

11. Bautin, N. N. and Cherniagin, B. M., A theoretical and experimental study of the dependence of the dynamic characteristics of a marine chronometer on the position of the balance spiral shoe. Izv. Akad. Nauk SSSR, OTN, Mekhanika i Mashinostroenie, № 2, 1963.

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ON THE FLOW OF A HEAVY FLUID IN A CHANNEL WITH A CURVILINEAR FLOOR

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Constructive proof is given of the single-valued solvability of the problem of flow of a heavy fluid with free surface in a channel with a curvilinear floor at fairly high Froude numbers and certain restrictions imposed on the floor shape. Nonconstructive proofs of the existence of solution with other restrictions on the floor shape were previously obtained in [1 — 3]. The proof of solvability in the case when the Froude number is reasonably close to but greater than unity appears in [4].

1. The stabilized flow of a perfect incompressible ponderable fluid bounded from above by free surface L and by a curvilinear floor S with horizontal asymptotes is considered in the plane $z = x + iy$ (Fig. 1). The coordinate origin is located on S with the y-axis directed vertically upward. At infinity upstream (to the left) the fluid flow is uniform and is defined by velocity V_0 and depth H of the stream.

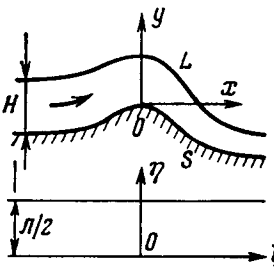

Fig. 1

Let l be the curvilinear abscissa of a point on S measured from the coordinate origin in the direction of flow, and β be the angle between the tangent to S in the direction of flow and the x-axis. We specify the shape of curve S by the equation

$$\beta = F(t), \quad t = l / H \quad (-\infty < t < \infty)$$

We assume function $F(t)$ to be twice differentiable and to satisfy for $-\infty < t < \infty$ the conditions

$$|F(t)| \leqslant B_0 e^{-b_0|t|}, \quad |F'(t)| \leqslant B_1 e^{-b_1|t|} |t|^{-1}$$
$$|F'(t)| \leqslant B_2, \quad |F''(t)| \leqslant B_3 |t|^{-1} \tag{1.2}$$

where B_0, B_1, B_2, B_3, b_0 and b_1 are some positive constants.

Let the band $K = \{0 < \eta < \pi / 2\}$, conformally represent the flow region in the plane of the auxiliary variable $\zeta = \xi + i\eta$, with the straight lines $\eta = \pi / 2$ and $\eta = 0$ corresponding, respectively, to the free surface and to the solid boundary, and point $\zeta = 0$ to the coordinate origin of plane z. The complex flow potential w is defined by formula

$$w = 2V_0 H \pi^{-1} \zeta \tag{1.3}$$